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Stability of classical chaotic motion under a system's perturbations

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We study in detail the time behavior of classical fidelity for chaotic systems. We show, in particular, that the asymptotic decay, depending on system dynamical properties, can be either exponential, with a rate determined by the gap in the discretized Perron-Frobenius operator, or algebraic, with the same power as for correlation functions decay. Therefore the decay of fidelity is strictly connected to correlations decay.

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As it is known, the exponential separation of orbits starting from slightly different initial conditions has been associated with classical chaos. It has been noticed that the situation in quantum mechanics is drastically different. Indeed the scalar product of two states $\langle \psi_1 | \psi_2 \rangle$ is time independent. This has led to the introduction of fidelity as a measure of the stability of quantum motion [1]. More precisely one considers the overlap of two states which, starting from the same initial conditions, evolve under two slightly different Hamiltonians H_0 and $H_\epsilon = H_0 + \epsilon V$. The fidelity is then given by $f(t) = |\langle \psi | \exp(iH_\epsilon t/\hbar) \exp(-iH_0 t/\hbar) | \psi \rangle|^2$. The quantity $f(t)$ can be seen as a measure of the so-called Loschmidt echo: a state $|\psi\rangle$ evolves for a time t under the Hamiltonian H_0, then the motion is reversed and evolves back for the same time t under the Hamiltonian H_ϵ and the overlap with the initial state $|\psi\rangle$ is considered.

However, we would like to stress that, in principle, such difference between classical and quantum mechanics actually does not exist. The Liouville equation, which describes classical evolution, is unitary and reversible as the Schrödinger equation. However, as stressed in several occasions (see, e.g., Ref. [2]), there exist time scales up to which quantum motion can share the properties of classical chaotic motion including local exponential instability. Due to the existence of such time scales, what may be different, and indeed it is, is the degree of the stability of dynamical motion. Indeed, as clearly illustrated in the analysis of Loschmidt echo in Ref. [3], quantum motion turns out to be more stable than the classical motion.

The growing interest in quantum computers has attracted recent interest in this quantity as a measure of the stability of quantum computation in the presence of hardware imperfections or noisy gate operations. Confining ourselves to classically chaotic systems, the emerging picture which results from analytical and numerical investigations [4–12] is that both exponential and Gaussian decays are present in the time behavior of fidelity. The strength of the perturbation determines which of the two regimes prevails. The decay rate in the exponential regime appears to be dominated either by the classical Lyapunov exponent or, according to the Fermi golden rule, by the spreading width of the local density of states.

In addition, at least for short times, the decaying behavior depends on the initial state (coherent state, mixture, etc.). While it can be true that, for practical purposes, the short time behavior of fidelity may be the most interesting one, it is also true, without any doubt, that in order to have a clear theoretical understanding and identify a possible universal type of quantum decay one needs to consider the asymptotic behavior of fidelity.

There arises the problem of understanding the corresponding classical decay of fidelity and later on inquiring about the time scales at which quantum decay mimics the classical one. In the present paper, we concentrate our attention on the classical behavior.

What do we know about the decay of classical fidelity for chaotic systems? What is the relation with correlation functions? Can we derive the decay of fidelity from the behavior of correlations or is fidelity a completely independent function? In a recent paper [12] it has been found numerically that, after an initial transient, classical fidelity decays exponentially and the rate is given by the Lyapunov exponent (see also Refs. [8,13]). This is also in agreement with previous papers [4–6] indicating that quantum fidelity, for strong enough quantum perturbation (which, for a fixed classical perturbation strength, corresponds to semiclassical region), decays exponentially with a rate given by the the Lyapunov exponent of the corresponding classical system. On the other hand, we know that the decay of correlation functions is not ruled by the Lyapunov exponent. In the first place, there is the general phenomenon of long time tails which means power law decay. In addition, for the special cases in which one can prove exponential decay, the rate is determined by the gap in the discretized Perron-Frobenius operator and not by the Lyapunov exponent.

In this paper, we show that the asymptotic decay of classical fidelity for chaotic systems is not related to the Lyapunov exponent: Similarly to correlation functions, this decay can be either exponential or power law. In the first case, the decay rate is determined by the gap in the discretized Perron-Frobenius operator, and in the latter case the power law has the same exponent as for correlation functions.

The classical fidelity $f(t)$ is defined as follows:

$$f(t)=\int_{\Omega}d\mathbf{x}\rho_{\epsilon}(\mathbf{x},t)\rho_0(\mathbf{x},t), \tag{1}$$

where the integral is extended over the phase space, and

$$\rho_0(\mathbf{x},t)=U_0^t\rho(\mathbf{x},0), \quad \rho_{\epsilon}(\mathbf{x},t)=U_{\epsilon}^t\rho(\mathbf{x},0) \tag{2}$$

give the evolution after t steps of the initial density $\rho(\mathbf{x},0)$ [assumed to be square normalized, i.e., $\int d\mathbf{x}\rho^2(\mathbf{x},0)=1$] as determined by the tth iteration of the Perron-Frobenius perators U_0 and U_{ϵ}, corresponding to the Hamiltonians H_0 and H_{ϵ}, respectively. The above definition can be shown to correspond to the classical limit of quantum fidelity [8,10]. In the ideal case of perfect echo ($\epsilon=0$), the fidelity does not decay, $f(t)=1$. However, due to chaotic dynamics, when $\epsilon\neq 0$ the classical fidelity decay sets in after a time scale

$$t_{\nu}\sim\frac{1}{\lambda}\ln\left(\frac{\nu}{\epsilon}\right), \tag{3}$$

required to amplify the perturbation up to the size ν of the initial distribution, with λ the Lyapunov exponent. Thus, for $t\gg t_{\nu}$ the recovery of initial distribution via the imperfect time-reversal procedure fails, and the fidelity decay is determined by the decay of correlations for a system that evolves forward in time according to the Hamiltonians H_0 (up to time t) and H_{ϵ} (from time t to time $2t$). This is conceptually similar to the "practical" irreversibility of chaotic dynamics: due to the exponential instability, any amount of numerical error in computer simulations rapidly effaces the memory of the initial distribution [3]. In the present case, the coarse graining which leads to irreversibility is not due to roundoff errors but due to a perturbation in the Hamiltonian.

In the following, we illustrate this general phenomenon in standard models of classical chaos, characterized by uniform exponential instability (the sawtooth map), marginal stability (the stadium billard), or mixed phase space dynamics (the kicked rotator).

The sawtooth map is defined by

$$\overline{p}=p+F_0(\theta), \quad \overline{\theta}=\theta+\overline{p}, \tag{4}$$

where (p,θ) are conjugated action-angle variables, $F_0=K_0(\theta-\pi)$, and the overbars denote the variables after one map iteration. We consider this map on the torus $0\leqslant\theta<2\pi$, $-\pi L\leqslant p<\pi L$, where L is an integer. For $K_0>0$ the motion is completely chaotic and diffusive, with the Lyapunov exponent given by $\lambda=\ln[(2+K_0+[(2+K_0)^2-4]^{1/2})/2]$. For $K_0>1$ one can estimate the diffusion coefficient D by means of the random phase approximation, obtaining $D\approx(\pi^2/3)K_0^2$. In order to compute fidelity (1), we choose to perturb the kicking strength $K=K_0+\epsilon$, with $\epsilon\ll K_0$. In practice, we follow the evolution of 10^8 trajectories, which are uniformly distributed inside a given phase space region of area A_0 at time $t=0$. The fidelity $f(t)$ is given by the percentage of trajectories that return back to that region after t iterations of map (4) forward, followed by the backward evolution, now with the perturbed strength K, in the same time interval t. In order to study the approach to equilibrium for fidelity, we consider the quantity $g(t)=[f(t)-f(\infty)]/[f(0)-f(\infty)]$; in this way $g(t)$ drops from 1 to 0 when t goes from 0 to ∞. We note that $f(0)=1$, while for a chaotic system $f(\infty)$ is given by the ratio A_0/A_c, with A_c the area of the chaotic component to which the initial distribution belongs.

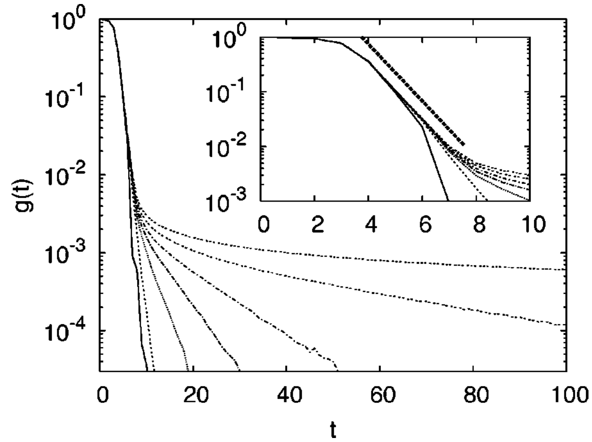

FIG. 1. Decay of the fidelity $g(t)$ for the sawtooth map with the parameters $K_0=(\sqrt{5}+1)/2$ and $\epsilon=10^{-3}$ for different values of $L=1,3,5,7,10,20,\infty$ from the fastest to the slowest decaying curve, respectively. The initial phase space density is chosen as the characteristic function on the support given by the $(\theta,p)\in[0,2\pi)[-\pi/100,\pi/100]$. Note that between the Lyapunov decay and the exponential asymptotic decay there is a $\propto 1/\sqrt{t}$ decay, as expected from the diffusive behavior. Inset: magnification of the same plot for short times, with the corresponding Lyapunov decay indicated as a thick dashed line.

The behavior of $g(t)$ is shown in Fig. 1, for $K_0=(\sqrt{5}+1)/2$ and different L values. One can see that only the short time decay is determined by the Lyapunov exponent. It takes place for $t_{\nu}<t<t_{\epsilon}$, with t_{ν} defined in Eq. (3) and $t_{\epsilon}\sim(1/\lambda)\ln(2\pi/\epsilon)$ time scale required to amplify the effect of the Hamiltonian perturbation up to the maximum extension in the angle θ. The Lyapunov regime is followed by a power law decay [12] $\propto 1/\sqrt{Dt}$ until the diffusion time $t_D\sim L^2/D$ and then the asymptotic relaxation to equilibrium takes place exponentially, with a decay rate γ (shown in Fig. 2), which, as discussed below, is ruled not by the Lyapunov exponent but by the largest Ruelle-Pollicott resonance [14]. In particular, it is ϵ independent.

We determine numerically these resonances for the sawtooth map using the following method [15,16].

(i) The phase space torus ($0\leqslant\theta<2\pi$, $-\pi L\leqslant p<\pi L$) is divided into $N\times NL$ square cells.

(ii) The transition matrix elements between cells are determined numerically by iterating for one map step the phase space distributions given by the characteristic functions of each cell: in this way we build a finite dimensional approximation of the one-period evolution operator U_0.

(iii) This truncated evolution matrix $U_0^{(N)}$ (of size $LN^2\times LN^2$) is diagonalized: it is no longer unitary, and its eigenvalues $z_i^{(N)}$ are inside the unit circle in the complex plane (an example is shown in Fig. 3). The nonunitarity of the coarse-grained evolution is due to the fact that the transfer of prob-

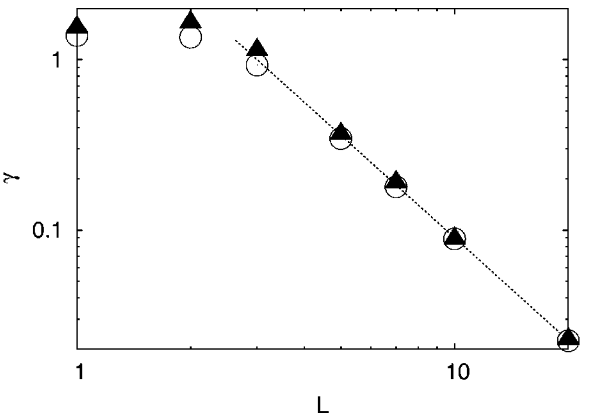

FIG. 2. Asymptotic exponential decay rates of fidelity for the sawtooth map $[K_0=(\sqrt{5}+1)/2,\ \epsilon=10^{-3}]$ as a function of L. The rates are extracted by fitting the tails of the fidelity decay in Fig. 1 (triangles) and from the discretized Perron-Frobenius operator (circles). The line denotes the $\propto 1/L^2$ behavior of the decay rates, predicted by the Fokker-Planck equation, which describes the classical motion in the diffusive regime.

ability to finer scale structures in the phase space is cutoff, and this results in an effective dissipation [15].

(iv) Resonances correspond to ''frozen'' nonunimodular eigenvalues, namely, $z_i^{(N)} \to z_i$ when $N\to\infty$, with $|z_i|<1$. Convergence of eigenvalues to values inside the unit circles comes from the asymptotic self-similarity of chaotic dynamics [15].

As it is known, the asymptotic ($t\to\infty$) relaxation of correlations is determined by the resonance with largest modulus, $|\tilde{z}|=\max_i|z_i|<1$, giving a decay rate $\gamma_0=\ln|\tilde{z}|$. In Fig. 2, we illustrate the good agreement between the asymptotic decay rate of fidelity (extracted from the data of Fig. 1) and the decay rate as predicted by the gap in the discretized Perron-

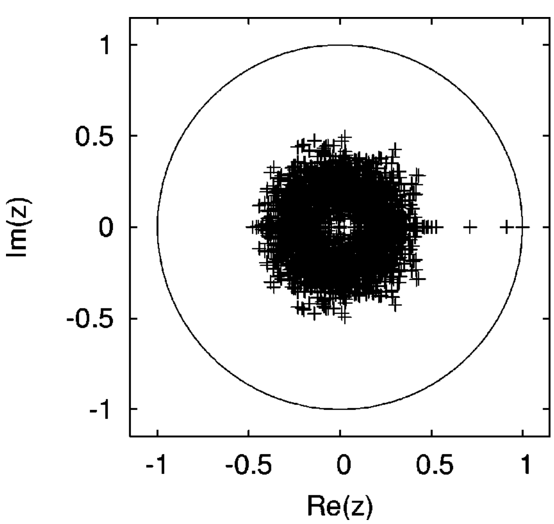

FIG. 3. Spectrum of the discretized Perron-Frobenius operator for the sawtooth map with parameters $K_0=(\sqrt{5}+1)/2$, $L=7$, and discretization $N=20$. The asymptotic decay of fidelity is determined by the largest modulus eigenvalue apart from the eigenvalue 1.

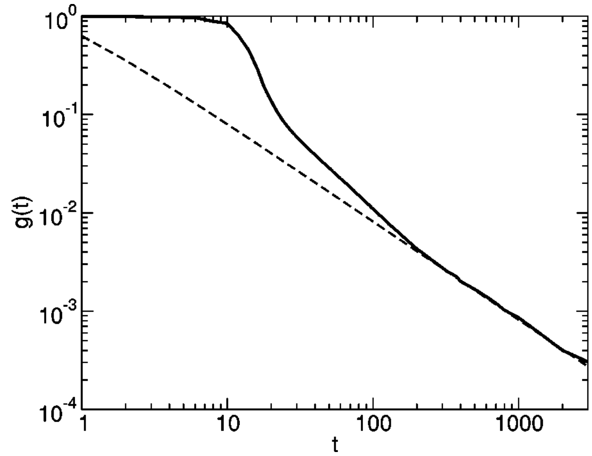

FIG. 4. Power law decay of fidelity for the stadium billiard with radius $R=1$ and the length of the straight segments $d_0=2$ (the perturbed stadium has $d=d_0+\epsilon$, with $\epsilon=2\times10^{-3}$). The initial phase space density was chosen to be a direct product of a characteristic function on a circle in configuration space, the center of which was at (0.5,0.25) as measured from the center of the billiard and its radius was 0.1, while for momenta the $\delta(|\mathbf{p}|-1)$ distribution was used. The dashed line has slope -1.

Frobenius spectrum. It should be stressed that, since fidelity involves forward and backward evolutions, the fidelity decay at time t has to be compared with the correlations decay at time $2t$. For this reason in Fig. 2 the circles correspond to $\gamma=2\gamma_0$.

We would like to stress that the same qualitative behavior of Fig. 1 is obtained in the presence of stochastic noise, e.g., the backward evolution is driven by a kicking strength $K(t)=K_0+\epsilon(t)$, with $\{\epsilon(t)\}_{t=1,2,\ldots}$ uniformly and ran-

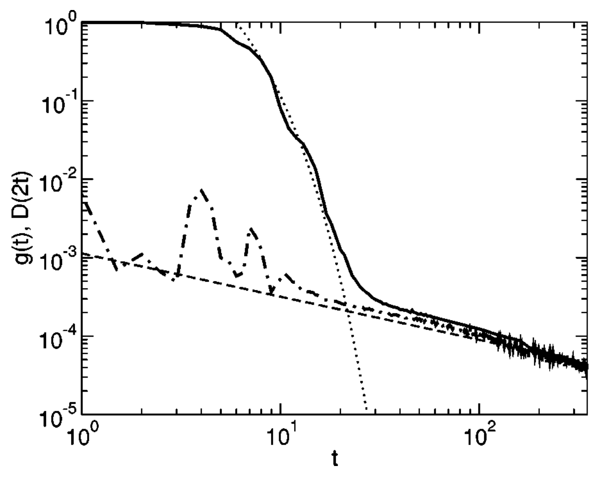

FIG. 5. The decay of fidelity for the kicked rotator with $K_0=2.5$, $L=1$, and $\epsilon=10^{-3}$ (full curve). The support of the initial (characteristic) density is inside the chaotic component, with $(\theta,p)\in[0,0.2][0,0.2]$. The dotted curve represents the exponential decay at a rate given by the Lyapunov exponent $\lambda\approx0.534$. The dashed line has slope -0.55. The dot-dashed curve gives the correlation decay $D(2t)$, for the same initial density and for twice the time t. It is clearly seen that, asymptotically, fidelity and correlations have the same power law decay $\propto t^{-0.55}$.

domly distributed inside the interval $[-\epsilon,\epsilon]$. In particular, we observed the initial Lyapunov decay and the asymptotic exponential relaxation with the same rate γ. This means that the effect of a noisy environment on the decay of fidelity for a classical chaotic system is similar to that of a generic static Hamiltonian perturbation.

Further confirmation for the validity of the above illustrated scenario has been obtained by analyzing systems in which the asymptotic decay of correlations is algebraic. This happens in the following cases.

(i) When the system possesses marginally stable orbits: A typical example is the stadium billiard in which, as it is known [17], correlations decay as $1/t$.

(ii) When there is mixed phase space [18]: A typical example is the kicked rotator model [described by Eq. (4) with $F_0 = K_0 \sin\theta$].

Since in the long time limit the fidelity decay at time t is still related to the decay of correlations at time $2t$, in the case of power law decay of correlations as $t^{-\alpha}$, we expect a power law decay of fidelity with the same exponent α. This is indeed confirmed by our numerical results. In Fig. 4 it is shown that, for the stadium billiard, fidelity decays asymptotically $\propto 1/t$, as expected. In Fig. 5, we compare the fidelity decay (at time t) and the correlations decay (at time $2t$) for the kicked rotator with kicking parameter $K_0 = 2.5$, for which the phase space contains chaotic components and stable islands. The correlator is given by $D(t) = [C(t) - C(\infty)]/[C(0) - C(\infty)]$, with $C(t) = \int_\Omega d\mathbf{x}\rho_0(\mathbf{x},t)\rho(\mathbf{x},0)$. It is seen that, after an initial Lyapunov decay, fidelity approaches the same asymptotic power law decay of correlations [19].

In summary, we have shown that in chaotic systems the asymptotic decay of classical fidelity, which describes the structural stability of motion under system's perturbations, is analogous to the asymptotic decay of correlation functions. This asymptotic decay can be either exponential or algebraic, depending on the dynamical properties of the system. In any instance, it is not related to the local exponential instability ruled by the Lyapunov exponent and it is ϵ independent. It would be interesting to understand what are the implications of these findings for the decay of quantum fidelity.

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[19] We note that the short time Lyapunov decay of fidelity is by no means a typical feature of correlation functions, since the short time decay of $D(t)$ is determined by the motion of the ''center of mass'' of the initial distribution $\rho(\mathbf{x},0)$, a trivial effect which is suppressed in fidelity due to the backward evolution.